ALGEBRAIC SURFACES FOR CAD AND COMPUTER GRAPHICS

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ABSTRACT

Most commercial CAD systems use parametric representations (e.g., NURBS) for defining free-form shapes. This leads to significant problems in certain types of geometric computation, and also in the exchange of CAD models between different systems. The paper suggests that there are good reasons for developing alternative approaches to the representation of free-form shapes in CAD, based on the use of algebraic surfaces. Work in this direction is currently at an early stage, but some fruitful directions are being identified. The paper surveys the current state of this work, suggests possible applications for algebraic surfaces in CAD and computer graphics, and identifies some fruitful research areas.

Keywords: algebraic surfaces, implicit surfaces, computer graphics, CAD.

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INTRODUCTION

All major commercial CAD systems provide capabilities for defining what is known as free-form surface geometry, which is needed for the modelling of such artefacts as shoes, ships, cars, aircraft and plastic supermarket bottles. Such systems all implement either non-uniform rational B-spline (NURBS) [PT95] or Bézier [Far93] representations, both of which are *parametric* in nature, representing points on a surface by relations of the form

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)], \qquad (1)$$

where x, y, z are cartesian coordinates, $\mathbf{r}$ is the position vector of a point on the surface and u, v may be thought of as defining a curvilinear coordinate system embedded in the surface [FP79]. However, the use of this type of geometry leads to certain problems:

- the computation of intersection curves is complex and error-prone;
- NURBS, in particular, can sometimes give non-intuitive results from standard constructional procedures, such as the propagation of sharp crease lines into surface interiors when smooth boundaries are interpolated;
- a given surface may be parametrized in infinitely many ways, and automatically generated parametrizations may be bad for application purposes (e.g., in manufacturing by numerically controlled machining).

An alternative form of surface representation makes use of *implicit* surfaces, represented by equations of the form

$$f(x,y,z) = 0. \qquad (2)$$

An implicit surface for which the function f is a polynomial is said to be *algebraic*. There is an extensive literature of algebraic surfaces, dating back well over one hundred years. The investigation of implicit surfaces in the wider sense is more recent. Bloomenthal [Blo97] has edited a book on the topic, which is oriented mainly towards applications in computer graphics and animation. It is suggested in this paper that some of the problems arising from the use of parametric surface representations in CAD can potentially be overcome by using implicit surfaces instead — algebraic surfaces, in particular.

In fact the parametric surfaces used in CAD usually have corresponding algebraic forms, and some of the problems arising in their use may be explained by examining the connection between the two types of representation. Parametric surfaces in CAD are generally of the polynomial or rational type, in which the functions $x(u,v), y(u,v), z(u,v)$ in Eq (1) have polynomial or rational form respectively. Suppose these functions are of degree n in u and v. Then the parametric surface has a corresponding algebraic representation of degree $2n^2$. Geometrically, this means that a given straight line may intersect the surface in as many as $2n^2$ points, so that the algebraic degree gives some measure of the possible undulation of the surface[1].

Now consider the fact that CAD systems routinely use surfaces with parametric degree 15×15 (giving algebraic degree 450) and that at least one system goes as high as 20×20 (giving algebraic degree no less than 800). Such surfaces are only used in tightly constrained situations, where many degrees of freedom are necessary to meet the constraints. However, the equation of a general algebraic surface of degree N contains $\frac{1}{6}(N+1)(N+2)(N+3)$ coefficients, of which one may be chosen arbitrarily, the rest each representing one degree of geometric freedom. It is hard to believe that any engineering situation demands the phenomenal number of degrees of freedom given by this expression with $N = 450$, let alone $N = 800$.

It is also instructive to consider the geom-

[1] This is the case when there are two parameters and the patches are four-sided. Triangular patches may also be defined, having three parameters with a linear relationship among them; for such surfaces the corresponding algebraic degree is n^2.

etry of intersection curves. Two surfaces of algebraic degree N intersect in a curve of degree N^2 — this is a measure of oscillatory tendency, since it represents the number of times the curve may intersect with a plane. Thus, two surfaces having rational parametric degree $n \times n$ intersect in a curve of algebraic degree $4n^4$. For a bicubic parametric surface this degree is 324; for a CATIA surface of parametric degree 15×15 it is more than 2×10^5.

Thus it appears that current CAD systems are using inappropriate types of geometry. The very high algebraic degree of such surfaces, and their associated oscillatory tendency, makes it very difficult to devise geometric algorithms that are not only robust but also fast and accurate. The structural integrity of boundary representation (Brep) solid models, in particular, can suffer because of this difficulty. Errors in intersection computations can cause the geometry of the model to be incompatible with its associated topological (connectivity) information. Such problems have been highlighted in recent trials of the CAD model exchange standard ISO 10303 (STEP) [Int94]; exchanges can be incomplete or can fail altogether because of such discrepancies in the transmitted model.

The suggestion naturally arises, therefore, that it may be possible to achieve the necessary flexibility for free-form design, while also achieving better modeling robustness, through the use of algebraic surfaces of comparatively low degree. The major difficulty, as will be seen, lies in finding a suitable means for controlling the shape of algebraically defined surfaces.

At present there is little motivation to turn to the use of *non-rational* parametric surfaces or non-algebraic implicit surfaces for CAD applications. The reason for this is that rational functions[2] are the most general functions that can be evaluated using the four basic computational operations of addition, subtraction, multiplication and division; the use of any other functions would introduce inevitable approximation errors in the evaluation of surface geometry. However, non-algebraic implicit surface representations are becoming popular for use in computer graphics and animation [Blo97].

[2] Here and in what follows, rational functions will be taken to include polynomials.

CHANGE OF REPRESENTATION

Some facts regarding the relation between algebraic and parametric surfaces will be briefly related here. First, any parametric surface with a rational parametrization corresponds to an algebraic surface, as mentioned above. The process of deriving the algebraic from the parametric form is called *implicitization*. Several methods for this have been investigated, mostly based on the use of resultants, but they are only practical for parametric surfaces of low degree, because otherwise the resulting algebraic surfaces will have immensely high degree [SAG84, SAG85].

On the other hand, not all algebraic surfaces possess a rational parametrization, though they may have parametrizations expressible in terms of fractional powers or transcendental functions. The inverse process to implicitization, called *parametrization*, is thus in general not possible if a rational parametric form is desired. In practice there is some virtue in working with surfaces that have both an algebraic and a parametric form, since the simplest approach to the calculation of intersections between two surfaces becomes possible when one is expressed implicitly and the other parametrically [PG86]. Intersection calculations are frequently required during the generation of a CAD model, and it is important that they can be performed quickly, robustly and accurately. This requires that algorithms are efficient in handling two types of complexity:

global complexity – the intersection curve may consist of numerous disjoint branches, some of which can be small closed loops , difficult to detect.

local complexity – the curve may exhibit singularities such as cusps, self-intersections, self-tangencies etc. These can all confuse the kind of 'marching' algorithms usually used in practice for tracing intersections.

It should be noted that the higher the algebraic degree of the interescting surfaces, the more likely it is that these problems will occur.

ALGEBRAIC SURFACES IN CAD

The Current Situation

All CAD systems implement the following 'simple' surfaces: the plane, right circular cylinder, right circular cone, sphere and torus. The plane has a linear equation, and all the others except for the torus are *quadric* surfaces for which f in Eq (2) is quadratic. The torus is a *quartic* surface of algebraic degree 4. A few systems provide more general quadrics (e.g., elliptic cones) and more general quartics, in particular the Dupin cyclides, of which more later.

Although these are all algebraic surfaces they are not usually represented in CAD systems in their algebraic forms, though these may be generated if required for certain types of calculation. Typically such a surface is represented either in a NURBS form or in terms of what is called its *geometric* representation, in terms of a small number of shape-defining entities. Thus a plane is often defined in terms of a point and a normal direction, a cylinder in terms of a centre-line and a radius, a cone in terms of an apex point, a centre-line and an angle, and a sphere in terms of a centre-point and a radius. The torus is characterized by a centre-point, a centre-line and major and minor radii.

One of the problems of working with more general algebraic surfaces is that it is difficult to identify such convenient geometric 'handles' for their manipulation. This is where the NURBS and Bézier parametric surface representations do have a distinct advantage; they allow the design and modification of surface geometry in terms of 'control points' whose relation to the surface is fairly easy for the designer to comprehend. We will return to this matter of geometric control in a later section.

Choice of Surface Degree

If attention is focused on free-form design, overall surfaces will often be modeled in terms of patches of simpler surfaces joining with some prescribed level of geometric continuity. For some applications simple G^1 (tangent plane) continuity of adjoining patches will be adequate. For others, e.g. car body design, G^2 (surface curvature) continuity may be demanded. Such requirements will affect the choice of degree of the algebraic surfaces to be used. Most research to date has concentrated on the achievement of G^1 continuity. Even in this case there is so far no unanimity of opinion on the choice of degree giving the optimal compromise between geometric flexibility and intuitive designer control.

There is fairly general agreement that the quadric surfaces do not allow sufficient geometric freedom for freeform design, though there has recently been an investigation into their use for computer graphics modeling [FC97]. Cubic algebraic surfaces have been studied by Sederberg [Sed90]. He developed a means of controlling them for design, though not in a manner that would be intuitive for a non-mathematical designer. Sederberg also noted an inherent problem of algebraic surfaces in general; the topology of the surface can change significantly and unexpectedly as coefficients of the defining equation are varied.

From the formula given in the Introduction, the algebraic cubic has 19 degrees of freedom for design. This may be compared with 48 for the (non-rational) bicubic parametric patch, though this is not an exact comparison since those 48 define the patch boundary as well as the surface geometry they lie on.

Quartic Surfaces

Since the cubic surfaces appear to show little promise for CAD, attention has turned to the quartics. The general quartic algebraic surface has 34 degrees of freedom, and it is consequently difficult to provide intuitive means for its geometric control. The study of the quartics for use in CAD has therefore been restricted to special cases, some of which show promise for free-form design. Sederberg has examined the Steiner surfaces [SA85], which provide only limited geometric flexibility and are not easy to control. They have the added disadvantage that they naturally give rise to triangular patches, whereas quadrilateral patches are preferred in CAD. The Dupin cyclides [DMP93] are easier to handle from the designer's point of view, having a 'geometric' representation in terms of three shape-defining variables. They have been found useful for the construction of blend surfaces in solid modeling [Pra90, Pra95]. These cyclides give rise to quadrilateral surface patches, appropriate for CAD use, though Martin et al. [MdPS86] found that they give insufficient freedom for fully free-form design.

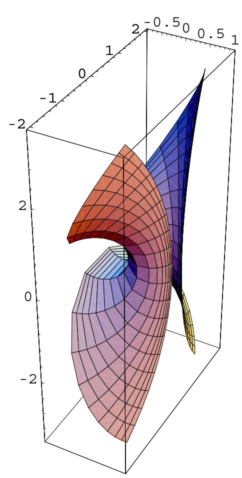

Figure 1: **A cubic Dupin cyclide patch**

More recently Degen, Pratt and others have been studying supercyclides, generalizations of the Dupin cyclides [Deg94, Pra96, Pra97]. These include projective transformations of the quartic Dupin cyclides and their degenerate cubic cases, and therefore permit greater freedom for shape modeling.

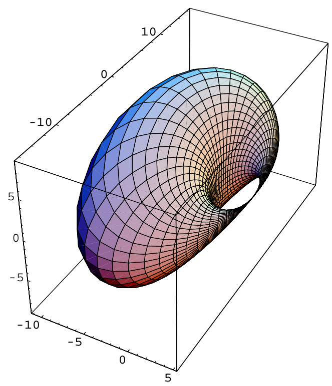

Figure 2: **A supercyclide (ellipside)**

The supercyclides can be parametrized by a conjugate net of conic curves, and they have the additional property that their tangent planes around any such curve envelope a quadric cone. In the special case of the Dupin cyclides the conic curves are circles and the tangent cones right circular cones. The existence of these cones makes it easy to join supercyclide patches with tangent continuity into larger surfaces defined in a piecewise manner — it is only necessary to match two supercyclides to the same cone along a shared conic curve. The blending of quadric surfaces in solid modeling can be based on the same principle [AD96, Deg94].

Supercyclides, unlike Dupin cyclides, may provide sufficient geometric freedom for fully free-form surface design. The fact that they generate quadrilateral patches with rational biquadratic parametrizations [Pra96, Pra97] also makes their use compatible with existing standards for CAD data transfer, such as ISO 10303 (STEP) [Int94, Owe93]. This standard does not provide a means for transferring triangular patch data, since there are no current CAD systems that use such patches to define free-form surfaces[3].

[3] Although some systems provide non-standard n-sided patches, with $n \neq 4$, for use in specialized situations [Mal98].

A further advantage for the CAD use of Dupin cyclides and supercyclides lies in the existence of exact solutions to the inverse parametrization problem, i.e., the determination of parameter values for a given cartesian point on a surface [Pra97]. This type of calculation is a frequent requirement in geometric computations for CAD. Supercyclides have the further property that patches bounded by isoparametric curves on them have coplanar corners. This makes it almost trivial to generate approximate representations to any desired accuracy in terms of planar quadrilateral facets. Such a characteristic is useful for many purposes, including the efficient generation of graphical renderings. It also has potential as the basis of a subdivision method for computing intersections between supercyclide surfaces.

An additional class of quartic surfaces that may prove useful in CAD is another family of generalized cyclides extensively studied in the 19th century by Darboux [Dar96]. These have recently arisen in some work by Paluszny and Boehm [PB98], and great scope exists for further investigation of their practical use in design.

It may be concluded that certain classes of quartic algebraic surfaces show good potential for application in CAD. Most research to date has focused on the supercyclides (including the Dupin cyclides), which fit very well into the context of existing CAD systems. These may prove sufficient for the modeling of shapes in which only G^1 surface continuity is required, though clearly a higher degree will be necessary for the achievement of higher orders of continuity in piecewise defined surfaces.

Triangular Patch Methods

When working with general algebraic surfaces rather than cyclides and their generalizations, it proves easier to work with triangular rather than quadrilateral patches. Several approaches have been described for the construction of piecewise surfaces from triangular algebraic patches [Baj93, DTS93, Guo91, Sed85]. However, these are oriented more towards the generation of surfaces interpolating point and tangent data rather than design in any more usual sense. Some of these methods use patches of mixed degree, from three through five. Menon [Men94] has shown how low-degree algebraic surfaces can be used in constructive solid geometry (CSG), by building volumes with curved three-sided outer faces onto an initial triangulated polyhedral approximation of the desired shape.

Pure Blending

Apart from the work already cited, many workers have studied the use of algebraic surfaces purely for blending purposes (see Rockwood [Roc97] and the references it contains). However, the intention of this paper is to highlight the potential for research that may lead to a more general shape design capability using algebraic surfaces, with consequent improvements in accuracy and robustness resulting from the elimination of the unwarranted degrees of freedom associated with parametric geometry.

CONCLUSIONS

Much work remains to be done before algebraic surfaces of cubic and higher degree become useful in the modeling of free-form shapes in CAD and computer graphics. However, this paper has shown that some of the groundwork has been laid, and has indicated several directions in which future research in this area may lead to valuable new capabilities. The reward for solving the problems of modeling with algebraic surfaces will be more robust and efficient geometric algorithms, which should help to overcome some of the difficulties encountered with today's CAD systems.

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